ON THE SINGLE SERVER RETRIAL QUEUE WITH BALKING[1]

J.R. ARTALEJO

Departamento de Estadistica e I.O., Facultad de Matematicas
Universidad Complutense de Madrid, 28040 Madrid, Spain

M.J. LOPEZ-HERRERO

Escuela Universitaria de Estadistica
Universidad Complutense de Madrid, 28040 Madrid, Spain

ABSTRACT

We are concerned with the M/G/1 retrial queue with balking. The ergodicity condition is first investigated making use of classical mean drift criteria. The limiting distribution of the number of customers in the system is determined with the help of a recursive approach based on the theory of regenerative processes. Many closed form expressions are obtained when we reduce to the M/M/1 queue for some representative balking policies.

Keywords: balking, hypergeometric series, $M/G/1$ queue, packet switching networks, regenerative process, retrials.

RÉSUMÉ

Dans cet article on étudie une file d'attente du type M/G/1 avec rappels et "balking". Premièrement on recherche la condition d'ergodicité en utilisant les critériums classiques des dérives moyennes. La distribution stationnaire du nombre de clients dans le système est determinée avec l'aide d'une approximation recoursive basée sur la théorie des processus régénératifs. En outre, pour un modèle du type M/M/1 des expressions explicites sont obtenues.

1. INTRODUCTION

This paper deals with a single server queue of the type $M/G/1$ in which customers balk with a general probability p_i, depending on the number of customers in the system upon arrival. In addition, a secondary input of repeated attempts associated to a group of primary blocked customers is also assumed.

In many queueing situations a customer on arrival may balk, so there exists a vast literature devoted to the design and applications of such models (see Grassmann (1974), Schellhaas (1983), Ikeda and Nishida (1988), Krishna Kumar et al. (1993), Abou-El-ata and Hariri (1995) and their references). Most queueing systems with balking deal with the case of classical waiting lines, i.e., the server is always aware of the presence of customers and immediately turns to a waiting customer when a service ends. However, there are queueing models in which an arriving customer who finds the server busy must leave the service area and join a group of unsatisfied customers called 'orbit'. This situation arises in telephone systems, in local area networks and in many daily life queueing models. The repeated attempts can be modelled according to an individual or a collective discipline depending on each particular application. A review of the main results and applications of retrial queues can be found in Yang and Templeton (1987), Falin (1990) and Falin and Templeton (1997).

Falin (1990) (section 13) described a model with non persistent primary customers which is in fact a retrial queue with constant balking probabilities. On the other hand, the case where repeated attempts are non persistent can be thought as a related model

[1]Recd. June 97; Revd. Feb. 98 and Aug. 99

with impatient customer, i.e., the balking assumption implies an automatic system abandonment upon arrival at the system whereas impatience means to take the abandonment decision after some random time.

Recently, Artalejo (1995) and Falin and Artalejo (1995) considered a modified $M/M/c$ retrial queue in which an arriving customer who finds all servers accessible for him busy, joins either the waiting line with probability p_i, depending on the number i of customers in the queue, or the orbit with probability $q_i = 1 - p_i$.

In the present paper we extend the analysis of queueing models allowing the simultaneous presence of balking and repeated attempts. To that end, we study an $M/G/1$ queue with balking probabilities depending on the number of customers in the system upon arrival. It should be noted that the analysis in closed form of retrial queues with general interarrival times or general interrepetition periods remains still open.

The rest of the paper is organized as follows. The mathematical model is described in section 2. A potential application related to the congestion control of buffers arising in packet switching networks is also discussed. In section 3, we investigate the ergodicity condition. The joint distribution of the server state and the orbit length in steady state is studied in section 4. In section 5, we concentrate on the model at Markovian level and consider several particular cases for the balking probabilities p_i.

2. MODEL DESCRIPTION

We consider a single server queueing system to which primary customers arrive according to a Poisson process with mean rate λ. On finding the server busy and i units in the orbit, an arriving customer joins the retrial group with probability q_{i+1}, i.e., the balking probability is $p_{i+1} = 1 - q_{i+1}$. Access from orbit to the server is governed by an exponential law with linear intensity $\alpha(1 - \delta_{0j}) + j\mu$, when the orbit size is $j \in \mathbb{N}$, where δ_{ab} is Kronecker's delta. That linear retrial policy (see Artalejo and Gomez-Corral (1997)) allows us to consider simultaneously the classical policy where $\alpha = 0$ (Yang and Templeton (1987), Falin (1990) and Falin and Templeton (1997)) and the constant retrial policy where $\mu = 0$ (Martin and Artalejo (1995) and its references). It is usual to employ the classical retrial policy to model subscribers' behaviour in telephone systems where the repeated attempts are made individually by each blocked customer (see Cohen (1957)). Both retrial policies have applications in computer networks where the retrial description follows a collective policy and the repeated attempts are made by a computational device (see Yang and Templeton (1987) and Choi et al. (1992)).

The services times are general with probability distribution function $B(t)$ ($B(0) = 0$), first moment β_1, and Laplace-Stieltjes transform $\beta(\theta)$. The input stream of primary arrivals, service times and intervals between successive repeated attempts are assumed to be mutually independent.

The system at time t can be described by the process $X(t) = (C(t), N(t), \xi(t))$, where $C(t)$ is 0 or 1 according to whether the server is free or busy, $N(t)$ is the number of unsatisfied customers in orbit at time t. When $C(t) = 1$, then $\xi(t)$ represents the elapsed time of the customer being served. In what follows, we neglect $\xi(t)$ and consider only the pair $Y(t) = (C(t), N(t))$ which state space is $S = \{0, 1\} \times \mathbb{N}$. The state space and the transitions among states are shown in Figure 1.

From the model description, it is clear that the evolution of our retrial queue is described in terms of an alternating sequence of idle and busy periods for the server. At any service completion epoch, the server becomes free. Next, the following customer who accesses to the service facility is determined by a competition between two exponential laws of rates λ and $\alpha(1 - \delta_{0j}) + j\mu$. It should be pointed out that this fact is the main difference with classical queueing systems without retrials.

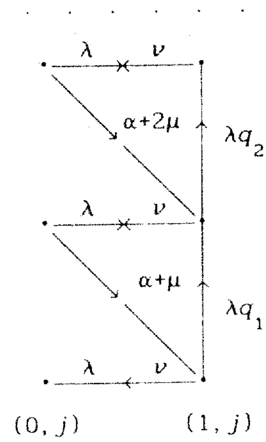

Figure 1: The state space and transitions

The above retrial queue with balking is a generalization of a few well known queueing systems. As a potential application we next examine the dynamic behaviour of a packet switching network with repeated attempts and control of the inner congestion.

The existing literature shows that retrial queues are an useful tool for the stochastic modelling of many computer networks. For the description of practical applications in packet switching networks, computer networks with star topology and local area networks operating under the so-called Carrier Sense Multiple Access protocol, the interested reader is referred to Yang and Templeton (1987), Choi et al. (1992), Khomichkov(1995), Janssens (1997) and the references therein.

In a general setting, a local area network provide communication paths among the computers and other devices connected to the nodes of the network. Due to economic reasons, the networks are designed to make possible the optimization of existing resources by sharing circuits. To that end, the nodes of a computer can be connected in a variety of ways. In this context, a major motivation for our model comes from a natural generalization of the packet switching network described by Yang and Templeton (1987). They considered a computer network in which there are a group of host computers connected to interface message processors. Messages arrive at the host computer following a Poisson stream. If the host computer wishes to transmit the message to another host computer, it must send the message and the final address to the interface message processor to which it is associated. If the processor is free the message is accepted; otherwise the message come back to the host computer and is stored in a buffer to be retransmitted some time later. The buffer in the host computer, the interface processor and the retransmission policy correspond to the orbit, the server and the retrial discipline, respectively, in the queueing terminology.

We now extend Yang and Templeton's description by introducing a link among the interface message processors and a mechanism to control the system congestion. One of the commonly used topologies for connecting the nodes of a computer network is the bus topology. The most important feature of the bus topology is that a transmitting station (interface processor in our application) sends the message through the bus to the rest of interface processors which are able to read destination addresses and select for complete reading those messages for which the interface processor is the appropriate destination. Information about the inner congestion in each host computer is also spread through the bus.

It should be noted that all interface message processors are sharing the same bus, so a routing policy is necessary to avoid collisions between two processors trying to transmit simultaneously their messages. However, in this paper we will restrict ourselves not to the global analysis of the whole network but only to a particular host computer and the processor at which it is connected. Finally, we introduce the balking probabilities as a mechanism to control the level of internal congestion in the buffer. In fact, we consider two possibilities depending on the available information coming from the bus:

i) *Partial information.* If the available information is reduced to the state of the buffer in the host computer and the state of the corresponding interface message processor, then we choose decreasing probabilities q_k. The expected effect is to moderate the waiting time in the buffer and reduce the number of collisions.

ii) *Global information.* We now assume that the bus transmits information about the buffer size of the different host computers. Then the particular station under study will have capacity to increase the probabilities q_k when the global buffer space in the rest of stations is low.

3. ERGODICITY CONDITION

We now study necessary and sufficient conditions for the system to be stable. To that end, we shall investigate the ergodicity of the embedded Markov chain at departure epochs which is the embedded Markov renewal process of the continuous time Markov process $X(t)$ (see Çinlar (1975)).

Let η_n be the time of the n-th departure. Observe that the sequence $Y_n = N(\eta_n+)$ forms a Markov chain, which is the embedded Markov chain for our queueing system. Furthermore, for convenience of presentation we shall use the following notations:

$$\rho = \lambda q \beta_1 \text{ and } \gamma = \lambda q \beta_1 (\lambda + \alpha)\alpha^{-1},$$

where $q = \lim_{k\to\infty} q_k$. The existence of this limit is a common assumption in practical applications (see for instance Haight (1957), Subba Rao and Jaiswal (1965) and Grassmann (1974)).

The ergodicity condition is studied in the following theorem:

Theorem 1 *Let assume that* $\lim_{k\to\infty} q_k = q$, *then:*

i) If $\alpha \geq 0$ *and* $\mu > 0$, *then* $\{Y_n\}_{n=1}^{\infty}$ *is ergodic if* $\rho < 1$. *It is not ergodic if* $\rho > 1$. *If* $\rho = 1$ *and* $\{q_k\}_{k=1}^{\infty}$ *decreases to* q *then* $\{Y_n\}_{n=1}^{\infty}$ *is not ergodic.*

ii) If $\alpha > 0$ *and* $\mu = 0$, *then* $\{Y_n\}_{n=1}^{\infty}$ *is ergodic if* $\gamma < 1$. *It is not ergodic if* $\gamma > 1$. *If* $\gamma = 1$ *and* $\{q_k\}_{k=1}^{\infty}$ *decreases to* q *then* $\{Y_n\}_{n=1}^{\infty}$ *is not ergodic.*

Proof:

First, observe that $\{Y_n\}_{n=1}^{\infty}$ satisfies the fundamental equation

$$Y_{n+1} = Y_n - B_{n+1} + V_{n+1}, \tag{1}$$

where V_n is the number of customers joining the orbit during the n-th service time and, $B_n = 1$ if the n-th customer in service comes from the orbit and $B_n = 0$ otherwise.

Note that $\{Y_n\}_{n=1}^{\infty}$ is irreducible and aperiodic. To investigate the ergodicity we shall employ Foster's criterion, which states that an irreducible and aperiodic Markov chain is ergodic if exists a non negative function $f(i), i \in I\!N$, and $\epsilon > 0$ such that the mean drift

$$\varphi_i = E[f(Y_{n+1}) - f(Y_n) \mid Y_n = i] \tag{2}$$

is finite for all $i \in I\!N$, and $\varphi_i \leq -\epsilon$ for all $i \in I\!N$ except perhaps a finite number. In our case, we choose the test function $f(i) = i$. Then, we obtain

$$\begin{aligned}\varphi_i &= E[V_{n+1} - B_{n+1} \mid Y_n = i] \\ &= (\lambda + \alpha(1-\delta_{0i}) + i\mu)^{-1} \\ &\quad \times [\lambda E_i[V_{n+1}] + (\alpha(1-\delta_{0i}) + i\mu)(E_{i-1}[V_{n+1}] - 1)] \quad i \in I\!N. \qquad (3)\end{aligned}$$

It should be noted that the subindex in the expected values represents the number of customers in orbit just at the beginning of the $(n+1)$-th service period.

The number of customers joining the retrial group during a service time is governed by a birth process with rates $\{\lambda q_i\}_{i=1}^{\infty}$. Alternatively, we can see this non-stationary Poisson process as a double stochastic Poisson process whose rate varies randomly (see Cox and Isham (1980)). Thus, taking into account that $\lim_{k\to\infty} q_k = q$, we get the following upper bound for any $\epsilon > 0$:

$$\varphi_i \leq \lambda(q+\epsilon)\beta_1 - (\alpha + i\mu)(\lambda + \alpha + i\mu)^{-1}, \quad i \geq i(\epsilon). \qquad (4)$$

Now we consider the case $\alpha \geq 0$ and $\mu > 0$. If $\rho < 1$ and $\epsilon \in \left(0, q(\rho^{-1} - 1)\right)$ then $\lim_{i\to\infty} \varphi_i \leq \lambda(q+\epsilon)\beta_1 - 1 < 0$. Therefore, the chain $\{Y_n\}_{n=1}^{\infty}$ is ergodic. The argument for the case $\alpha > 0$ and $\mu = 0$ is similar. In this second case, we must choose $\epsilon \in \left(0, q(\gamma^{-1} - 1)\right)$.

To study non-ergodicity we follow the Theorem 1 in Sennott et al. (1983) which states that we can guarantee non ergodicity if $\{Y_n\}_{n=1}^{\infty}$ satisfies Kaplan's condition, $\varphi_j < \infty$, for all $j \in I\!N$, and there is an index j_0 such that $\varphi_j \geq 0$, for $j \geq j_0$. Now we have the lower bound

$$\varphi_i \geq \lambda(q-\epsilon)\beta_1 - (\alpha + i\mu)(\lambda + \alpha + i\mu)^{-1}, \quad i \geq i(\epsilon). \qquad (5)$$

If we consider the case $\alpha > 0$ and $\mu = 0$, and choose $\epsilon \in \left(0, q(1 - \gamma^{-1})\right]$ then we have $\varphi_i \geq 0$, for $i \geq i(\epsilon)$. When $\mu > 0$, choosing $\epsilon \in \left(0, q(1-\rho^{-1})\right)$ we have $\lim_{i\to\infty} \varphi_i \geq \lambda(q-\epsilon)\beta_1 - 1 > 0$. Furthermore, Kaplan's condition is fulfilled because there is an index k such that $p_{ij} = 0$, for $j < i - k$, $i > 0$, where $P = (p_{ij})$ is the transition matrix associated to $\{Y_n\}_{n=1}^{\infty}$.

Finally, we consider the cases $\rho = 1$ and $\gamma = 1$. Now the system is ergodic or not ergodic depending on the way in which $\{q_k\}_{k=1}^{\infty}$ converges to q. We will discuss this situation in more detail in section 5. If we assume that $\{q_k\} \downarrow q$ we can easily observe that for $\alpha > 0$ and $\mu = 0$ we get

$$\varphi_i \geq \lambda q\beta_1 - \alpha(\lambda+\alpha)^{-1} = \alpha(\lambda+\alpha)^{-1}(\gamma - 1) = 0, \quad i \geq 1, \qquad (6)$$

and when $\mu > 0$ we find that

$$\varphi_i \geq \rho - (\alpha + i\mu)(\lambda + \alpha + i\mu)^{-1} > \rho - 1 = 0, \quad i \geq 1. \qquad (7)$$

This completes the proof. •

4. JOINT DISTRIBUTION OF THE SERVER STATE AND THE NUMBER OF CUSTOMERS IN ORBIT

For the process $Y(t)$ we define the probabilities $P_{ij}(t) = P\{C(t) = i, N(t) = j\}$ for $t \geq 0$, $(i,j) \in S$. Our main objective in this section is to develop a recursive scheme for computing the limiting probabilities

$$P_{ij} = \lim_{t\to\infty} P\{C(t) = i, N(t) = j\}, \text{ for } (i,j) \in S. \qquad (8)$$

In previous works (Kok (1984) and Artalejo (1994)), numerically tractable algorithms for computing the limiting distribution in single server queues with repeated attempts were developed. The derivation employs a recursive approach based on the theory of regenerative processes. Schellhaas (1986) and Tijms (1994) showed the usefulness of this method in a general class of queueing systems.

We next extend the methodology to get explicit expressions for $\{P_{ij}\}_{(i,j)\in S}$ when the input stream depends on the number of customers in the orbit.

Let a regeneration cycle be the time between two successive visits of the process $Y(t)$ to the state $(0,0)$. Then, the process $Y(t)$ is a regenerative process with embedded renewal process $\{T_i\}_{i=1}^{\infty}$, where T_i denotes the i-th regeneration cycle. We also introduce some random variables:

$T =$ the length of a cycle,

$T_{ij} =$ the amount of time in a cycle during which $Y(t) = (i,j)$, $(i,j) \in S$,

$N_j =$ the number of service completions in a cycle at which j customers are left behind in orbit, $j \geq 0$.

First, from the theory of regenerative processes, we have

$$P_{ij} = \frac{E[T_{ij}]}{E[T]}, \quad (i,j) \in S. \tag{9}$$

We now observe that the number of transitions from state $(0,j)$ is equal to the number of transitions into $(0,j)$ in a regeneration cycle $(0,T]$. Equating the corresponding expectations, we have

$$(\lambda + \alpha(1-\delta_{0j}) + j\mu)E[T_{0j}] = E[N_j], \quad j \geq 0. \tag{10}$$

Furthermore, the number of transitions at which the orbit size increases from j to $j+1$ equals the number of transitions at which the orbit size decreases from $j+1$ to j. Taking expectations, we obtain that

$$(\alpha + (j+1)\mu)E[T_{0,j+1}] = \lambda q_{j+1} E[T_{1j}], \quad j \geq 0. \tag{11}$$

Dividing (11) by $E[T]$ we have from (9) that

$$(\alpha + (j+1)\mu)P_{0,j+1} = \lambda q_{j+1} P_{1j}, \quad j \geq 0. \tag{12}$$

To obtain other relationships among the probabilities $\{P_{ij}\}_{(i,j)\in S}$, we define:

$A_{kj} =$ The expected amount of time that during a service time j customers are in orbit given that the previous service time left k customers in orbit.

Now a straightforward application of Wald's theorem yields

$$E[T_{1j}] = \sum_{k=0}^{j+1} E[N_k] A_{kj}, \quad j \geq 0. \tag{13}$$

From (9), (10) and (13) we get

$$\begin{aligned}(1 - \lambda q_{j+1}(1 + \lambda(\alpha + (j+1)\mu)^{-1})A_{j+1,j})P_{1j} \\ = \lambda A_{0j} P_{00} + (1-\delta_{0j}) \sum_{k=1}^{j} \lambda q_k (1 + \lambda(\alpha + k\mu)^{-1}) A_{kj} P_{1,k-1}, \quad j \geq 0. \end{aligned} \tag{14}$$

The above formula provides a stable scheme for computing the probabilities $\{P_{1j}\}_{j=0}^{\infty}$ in terms of P_{00}. The probabilities $\{P_{0j}\}_{j=0}^{\infty}$ can be obtained from (12). Finally, we can find P_{00} by using the normalization condition

$$P_{00} = 1 - \sum_{j=1}^{\infty} P_{0j} - \sum_{j=0}^{\infty} P_{1j}. \tag{15}$$

It remains to specify the calculation of the quantities A_{kj}. This will be done with the help of the following auxiliary quantity:

B_{kj} = the expected amount of time that during a service time j customers are in orbit given that at the beginning of the service k customers were in orbit.

By connecting A_{kj} and B_{kj} we obtain

$$A_{j+1,j} = (\alpha + (j+1)\mu)(\lambda + \alpha + (j+1)\mu)^{-1} B_{jj}, \quad j \geq 0, \tag{16}$$
$$A_{kj} = (\lambda + \alpha(1-\delta_{0k}) + k\mu)^{-1}[(\alpha(1-\delta_{0k}) + k\mu)B_{k-1,j} + \lambda B_{kj}], \ 0 \leq k \leq j \tag{17}$$

We now observe that an infinitesimal interval $(t, t+\Delta t)$ contributes to B_{kj} if: i) the service time has not been completed before time t, and ii) at time t there are j customers in orbit given that at time $t=0$ the orbit size was k.

Then, we have

$$B_{kj} = \int_0^{\infty} \mathcal{K}_{kj}(t)(1 - B(t))\, dt, \quad 0 \leq k \leq j, \tag{18}$$

where $\mathcal{K}_{kj}(t)$ is the probability that, starting from the state k, the birth process with rates $\{\lambda q_i\}$ is at the state j after time t.

The forward equations for $\mathcal{K}_{kj}(t)$ are given by:

$$\begin{aligned} \mathcal{K}_{kj}(0) &= \delta_{kj}, \\ \mathcal{K}'_{kk}(t) &= -\lambda_k\, \mathcal{K}_{kk}(t), \\ \mathcal{K}'_{kj}(t) &= -\lambda_j\, \mathcal{K}_{kj}(t) + \lambda_{j-1}\, \mathcal{K}_{k,j-1}(t), \quad j \geq k+1, \end{aligned} \tag{19}$$

where $\lambda_k = \lambda q_{k+1}$, for $k \geq 0$. It is known that the solution of (19) is given by

$$\mathcal{K}_{kj}(t) = \sum_{n=k}^{j} C_{nj}\, \lambda_n\, e^{-\lambda_n t}, \quad j \geq k, \tag{20}$$

where

$$C_{nj} = (\lambda_j)^{-1} \prod_{i=k,\ i\neq n}^{j} \lambda_i(\lambda_i - \lambda_n)^{-1}, \quad k \leq n \leq j. \tag{21}$$

It should be noted that (20) is valid when all the birth parameters $\{\lambda q_i\}$ are distinct (see Kulkarni (1995)); it is also assumed that the improper product $\prod_{i=k,\ i\neq n}^{k}$ is equal to one. The solution for a general sequence $\{\lambda q_i\}$ can be obtained following the lines described in Appendix I, Kleinrock (1975).

With the help of (20) and (21) we reexpress (18) as follows

$$B_{kj} = \sum_{n=k}^{j} C_{nj}\,(1 - \beta(\lambda_n)), \quad 0 \leq k \leq j. \tag{22}$$

Finally, we observe that the computation of P_{00} implies the truncation of the infinite series in formula (15). Thus, we must extend our arguments to a model with a finite orbit capacity, say $\mathbf{K}$. This finite retrial queue is a minor variant of the main model investigated earlier. It can be easily proved that B_{kj} are now given by

$$\begin{aligned} B_{kj} &= \sum_{n=k}^{j} C_{nj}\,(1-\beta(\lambda_n)), \quad 0 \le k \le j \le \mathbf{K}-1, \\ B_{k\mathbf{K}} &= \lambda_{\mathbf{K}-1} \sum_{n=k}^{\mathbf{K}-1} C_{n,\mathbf{K}-1}\,(\beta_1 - (\lambda_n)^{-1}(1-\beta(\lambda_n))), \quad 0 \le k \le \mathbf{K}-1, \\ B_{\mathbf{KK}} &= \beta_1, \end{aligned} \tag{23}$$

where C_{nj} were given in (21).

5. THE MODEL AT MARKOVIAN LEVEL

Through this section we take $B(t) = 1 - e^{-\nu t}$, $t \ge 0$, and assume that the ergodicity condition is fulfilled. It is easily verified that the limiting probabilities are subject to the equations

$$(\lambda + \alpha(1-\delta_{0j}) + j\mu)\,P_{0j} = \nu\,P_{1j}, \quad j \ge 0, \tag{24}$$

$$(\alpha + (j+1)\mu)\,P_{0,j+1} = \lambda\,q_{j+1}\,P_{1j}, \quad j \ge 0. \tag{25}$$

Then from (24) and (25) it follows that

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^{j} \prod_{k=1}^{j} q_k \frac{\left(\frac{\lambda+\alpha}{\mu}\right)_j}{\left(\frac{\alpha}{\mu}+1\right)_j} P_{00}, \quad j \ge 1, \tag{26}$$

$$P_{1j} = \left(\frac{\lambda}{\nu}\right)^{j+1} \prod_{k=1}^{j} q_k \frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_j}{\left(\frac{\alpha}{\mu}+1\right)_j} P_{00}, \quad j \ge 0, \tag{27}$$

where

$$(x)_n = \begin{cases} 1, & \text{if } n = 0, \\ x(x+1)\cdots(x+n-1), & \text{if } n \ge 1, \end{cases}$$

is the Pochammer symbol.

Now P_{00} is obtained by using the normalization condition (15), so we have

$$P_{00}^{-1} = \frac{\alpha}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\nu}\right)^{j} \prod_{k=1}^{j} q_k \frac{\left(\frac{\lambda+\alpha}{\mu}\right)_j}{\left(\frac{\alpha}{\mu}+1\right)_j} + \frac{\lambda}{\nu}\sum_{j=0}^{\infty}\left(\frac{\lambda}{\nu}\right)^{j} \prod_{k=1}^{j} q_k \frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_j}{\left(\frac{\alpha}{\mu}+1\right)_j}. \tag{28}$$

In the particular case $\mu = 0$ and $\alpha > 0$ the above expressions can be simplified as follows

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\prod_{k=1}^{j} q_k \left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^{j} P_{00}, \quad j \ge 1, \tag{29}$$

$$P_{1j} = \frac{\lambda}{\nu}\prod_{k=1}^{j} q_k \left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^{j} P_{00}, \quad j \ge 0, \tag{30}$$

$$P_{00}^{-1} = \frac{\alpha}{\lambda+\alpha} + \frac{\lambda(\lambda+\nu+\alpha)}{\nu(\lambda+\alpha)}\sum_{j=0}^{\infty}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^{j}\prod_{k=1}^{j} q_k. \tag{31}$$

Our second goal is to find the partial factorial moments of the limiting distribution $\{P_{ij}\}$. The partial factorial moments M_k^i, for $i \in \{0,1\}$ and $k \in \mathbb{N}$, are denoted as follows

$$M_0^i = \sum_{j=0}^{\infty} P_{ij}, \qquad M_k^i = \sum_{j=k}^{\infty} j(j-1)\cdots(j-k+1)\, P_{ij}, \quad k \geq 1,\ i \in \{0,1\}. \tag{32}$$

First, we consider the case $\mu > 0$. Utilizing that $(1)_j = j!$ and $(a)_{n+k} = (a)_k(a+k)_n$, we get after some manipulations that:

$$M_k^0 = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^k \frac{\left(\frac{\lambda+\alpha}{\mu}\right)_k k!}{\left(\frac{\alpha}{\mu}+1\right)_k} P_{00} \sum_{n=0}^{\infty}\left(\frac{\lambda}{\nu}\right)^n \frac{1}{n!}\prod_{i=1}^{n+k} q_i \frac{\left(\frac{\lambda+\alpha}{\mu}+k\right)_n (k+1)_n}{\left(\frac{\alpha}{\mu}+k+1\right)_n}, \quad k \geq 1, \tag{33}$$

$$M_k^1 = \left(\frac{\lambda}{\nu}\right)^{k+1} \frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_k k!}{\left(\frac{\alpha}{\mu}+1\right)_k} P_{00} \sum_{n=0}^{\infty}\left(\frac{\lambda}{\nu}\right)^n \frac{1}{n!}\prod_{i=1}^{n+k} q_i \frac{\left(\frac{\lambda+\alpha}{\mu}+k+1\right)_n (k+1)_n}{\left(\frac{\alpha}{\mu}+k+1\right)_n}, \quad k \geq 0, \tag{34}$$

$$M_0^0 = 1 - M_0^1. \tag{35}$$

Now the case $\mu = 0$ and $\alpha > 0$ reduces to the following expressions:

$$M_k^0 = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^k k!\, P_{00} \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^n \prod_{i=1}^{n+k} q_i, \quad k \geq 1 \tag{36}$$

$$M_k^1 = (\lambda+\alpha)\,\nu^{-1}\, M_k^0, \quad k \geq 1, \tag{37}$$

$$M_0^1 = \frac{\lambda}{\nu} P_{00} \sum_{n=0}^{\infty}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^n \prod_{i=1}^{n} q_i. \tag{38}$$

An easy application of the ratio test shows that M_k^i exists for every $k \in \mathbb{N}$ and $i \in \{0,1\}$.

Hereafter it is assumed that the probabilities q_k obey any of the following models:

Model 1. $q_k = k^{-1}$, $k \geq 1$.

The study of this model is motivated by the packet switching network operating under partial information described in section 2. Suppose that an arriving customer finds the server busy and k customers in orbit. If the orbit size increases then the waiting time is higher and the risk of a collision also increases. This fact have a discouraging effect and the customer is authorized to join the orbit with probability $(k+1)^{-1}$, i.e., the probability q_k is the inverse of the total number of customers in the system. In a more general setting we can assume that $\{q_k\}_{k=1}^{\infty}$ is a decreasing sequence or, indeed, that there exists an integer M, which is the greatest orbit size that the system will tolerate.

In this case the limiting probabilities and the factorial moments can be reexpressed in terms of hypergeometric functions.

Model 2. $q_k = k(k+p)^{-1}$, $k \geq 1$, $p \in \{1, 2, \cdots\}$.

The motivation for studying this model is double. From a theoretical point of view the analysis of model 2 allows us to complete the analysis of the ergodicity conditions given in section 3. The case in which the bus operates under the global information framework discussed in section 2 provides a potential application.

First, we study the model 1. Let us consider the *generalized hypergeometric series*

$$ {}_pF_q(a_1,\cdots,a_p;b_1,\cdots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{39}$$

and the *degenerate hypergeometric series* given by

$$\Phi(a;b;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}. \tag{40}$$

Observe that $\prod_{k=1}^{j} q_k=(j!)^{-1}$. Now the performance characteristics developed earlier can be rewritten in a more convenient form. These particular results can be summarized in the following theorem:

Theorem 2

i) If $\alpha\geq 0$, $\mu>0$ and $q_k=k^{-1}$, $k\geq 1$, then:

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^j\frac{\left(\frac{\lambda+\alpha}{\mu}\right)_j}{(1)_j\left(\frac{\alpha}{\mu}+1\right)_j}P_{00},\quad j\geq 1, \tag{41}$$

$$P_{1j} = \left(\frac{\lambda}{\nu}\right)^{j+1}\frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_j}{(1)_j\left(\frac{\alpha}{\mu}+1\right)_j}P_{00},\quad j\geq 0, \tag{42}$$

$$P_{00}^{-1} = \frac{\alpha}{\lambda+\alpha}+\frac{\lambda}{\lambda+\alpha}\Phi\left(\frac{\lambda+\alpha}{\mu};\frac{\alpha}{\mu}+1;\frac{\lambda}{\nu}\right)+\frac{\lambda}{\nu}\Phi\left(\frac{\lambda+\alpha}{\mu}+1;\frac{\alpha}{\mu}+1;\frac{\lambda}{\nu}\right), \tag{43}$$

$$M_k^0 = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^k\frac{\left(\frac{\lambda+\alpha}{\mu}\right)_k}{\left(\frac{\alpha}{\mu}+1\right)_k}P_{00}\,\Phi\left(\frac{\lambda+\alpha}{\mu}+k;\frac{\alpha}{\mu}+k+1;\frac{\lambda}{\nu}\right),\quad k\geq 1, \tag{44}$$

$$M_k^1 = \left(\frac{\lambda}{\nu}\right)^{k+1}\frac{\left(\frac{\lambda+\alpha}{\mu}+1\right)_k}{\left(\frac{\alpha}{\mu}+1\right)_k}P_{00}\,\Phi\left(\frac{\lambda+\alpha}{\mu}+k+1;\frac{\alpha}{\mu}+k+1;\frac{\lambda}{\nu}\right),\quad k\geq 0 \tag{45}$$

ii) If $\alpha>0$, $\mu=0$ and $q_k=k^{-1}$, $k\geq 1$, then:

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\frac{1}{j!}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^j P_{00},\qquad j\geq 1, \tag{46}$$

$$P_{1j} = \frac{\lambda}{\nu}\frac{1}{j!}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^j P_{00},\qquad j\geq 0, \tag{47}$$

$$P_{00}^{-1} = \frac{\alpha}{(\lambda+\alpha)}+\frac{\lambda(\lambda+\nu+\alpha)}{\nu(\lambda+\alpha)}e^{\lambda(\lambda+\alpha)/\nu\alpha}, \tag{48}$$

$$M_k^0 = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^k e^{\lambda(\lambda+\alpha)/\nu\alpha}P_{00},\qquad k\geq 1, \tag{49}$$

$$M_0^1 = \lambda\nu^{-1}e^{\lambda(\lambda+\alpha)/\nu\alpha}P_{00}, \tag{50}$$

$$M_k^1 = (\lambda+\alpha)\nu^{-1}M_k^0,\quad k\geq 1. \tag{51}$$

Now we turn our attention to model 2. First, we observe that $\{q_k\}_{k=1}^{\infty}$ increases to 1. Thus, the ergodicity for the cases *i)* $\mu > 0$ and $\rho = 1$, and *ii)* $\alpha > 0$, $\mu = 0$ and $\gamma = 1$ is not covered by Theorem 1. However, the continuous Markov chain $(C(t), N(t))$ is non-explosive so it is ergodic if and only if one can find a probability solution to the system (24) – (25). Taking into account that $\prod_{k=1}^{j} q_k = j!/(p+1)_j$, we reduce after some algebra the general expressions (26) – (31) and (33) – (38) to the following theorem.

Theorem 3

i) If $\alpha \geq 0$, $\mu > 0$ *and* $q_k = k(k+p)^{-1}$, $k \geq 1$, $p \in \{1, 2, \cdots\}$, *then*

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^j \frac{(1)_j\left(\frac{\lambda+\alpha}{\mu}\right)_j}{(p+1)_j\left(\frac{\alpha}{\mu}+1\right)_j} P_{00}, \quad j \geq 1, \tag{52}$$

$$P_{1j} = \left(\frac{\lambda}{\nu}\right)^{j+1} \frac{(1)_j\left(\frac{\lambda+\alpha}{\mu}+1\right)_j}{(p+1)_j\left(\frac{\alpha}{\mu}+1\right)_j} P_{00}, \quad j \geq 0, \tag{53}$$

$$\begin{aligned} P_{00}^{-1} &= \frac{\alpha}{\lambda+\alpha} + \frac{\lambda}{\lambda+\alpha}\, {}_3F_2\left(\frac{\lambda+\alpha}{\mu}, 1, 1; p+1, \frac{\alpha}{\mu}+1; \frac{\lambda}{\nu}\right) \\ &\quad + \frac{\lambda}{\nu}\, {}_3F_2\left(\frac{\lambda+\alpha}{\mu}+1, 1, 1; p+1, \frac{\alpha}{\mu}+1; \frac{\lambda}{\nu}\right), \end{aligned} \tag{54}$$

$$\begin{aligned} M_k^0 &= \frac{\lambda}{\lambda+\alpha}\left(\frac{\lambda}{\nu}\right)^k \frac{(k!)^2\left(\frac{\lambda+\alpha}{\mu}\right)_k P_{00}}{(p+1)_k\left(\frac{\alpha}{\mu}+1\right)_k} \\ &\quad {}_3F_2\left(\frac{\lambda+\alpha}{\mu}+k, k+1, k+1; p+k+1, \frac{\alpha}{\mu}+k+1; \frac{\lambda}{\nu}\right), \quad k \geq 1, \end{aligned} \tag{55}$$

$$\begin{aligned} M_k^1 &= \left(\frac{\lambda}{\nu}\right)^{k+1} \frac{(k!)^2\left(\frac{\lambda+\alpha}{\mu}+1\right)_k P_{00}}{(p+1)_k\left(\frac{\alpha}{\mu}+1\right)_k} \\ &\quad {}_3F_2\left(\frac{\lambda+\alpha}{\mu}+k+1, k+1, k+1; p+k+1, \frac{\alpha}{\mu}+k+1; \frac{\lambda}{\nu}\right), \quad k \geq 0 \end{aligned} \tag{56}$$

ii) If $\alpha > 0$, $\mu = 0$ *and* $q_k = k(k+p)^{-1}$, $k \geq 1$, $p \in \{1, 2, \cdots\}$, *then*

$$P_{0j} = \frac{\lambda}{\lambda+\alpha}\frac{(1)_j}{(p+1)_j}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^j P_{00}, \quad j \geq 1, \tag{57}$$

$$P_{1j} = \frac{\lambda}{\nu}\frac{(1)_j}{(p+1)_j}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^j P_{00}, \quad j \geq 0, \tag{58}$$

$$P_{00}^{-1} = \frac{\alpha}{\lambda+\alpha} + \frac{\lambda(\lambda+\nu+\alpha)}{\nu(\lambda+\alpha)}\, {}_2F_1\left(1, 1; p+1; \frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right), \tag{59}$$

$$\begin{aligned} M_k^0 &= \frac{\lambda}{\lambda+\alpha}\frac{(k!)^2}{(p+1)_k}\left(\frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right)^k P_{00} \\ &\quad {}_2F_1\left(k+1, k+1; p+k+1; \frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right), \quad k \geq 1, \end{aligned} \tag{60}$$

$$M_k^1 = (\lambda+\alpha)\nu^{-1}M_k^0, \quad k \geq 1, \tag{61}$$

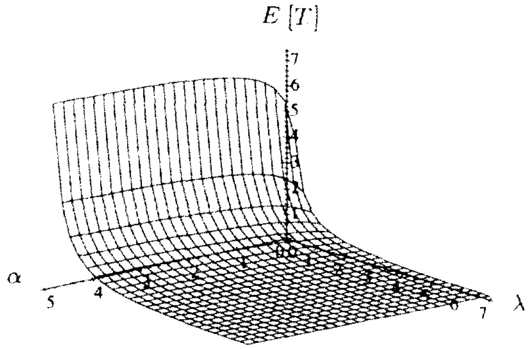

Figure 2: The effect of α and λ on $E[T]$.

$$M_0^1 = \frac{\lambda}{\nu} \, {}_2F_1\left(1, 1; p+1; \frac{\lambda(\lambda+\alpha)}{\nu\alpha}\right) P_{00}. \tag{62}$$

Finally, an application of ratio and Rabee's tests allows us to complete the discussion of the ergodicity condition. In this sense, we have

a) If $\alpha \geq 0$, $\mu > 0$ and $\rho = 1$, then the system is ergodic if and only if $\lambda < \mu(p-1)$. In addition, M_k^0 exists if and only if $k < p - \lambda/\mu$, and M_k^1 exists if and only if $k < p - 1 - \lambda/\mu$.

b) If $\alpha > 0$, $\mu = 0$ and $\gamma = 1$, then the system is ergodic if and only if $p \geq 2$. Further, M_k^0 and M_k^1 exist if and only if $k \leq p - 2$.

The main conclusion is that model 2 with increasing probabilities q_i provides a situation in which the ergodicity and the existence of the moment of order k (for the cases $\rho = 1$ and $\gamma = 1$) depend on constraints involving the system parameters λ, μ and p.

It should be noted that the main results of this section are given in terms of generalized and degenerate hypergeometric series. Thus numerical evaluation can be done with the help of well-known mathematical libraries such us MATHEMATICA and MAPLE.

We now illustrate the effect of the parameters on the main performance characteristics of model 2. Numerical solutions we obtained by using version 2.0a of Maple V.

In a first set of experiments we consider the constant retrial case $\alpha > 0$ and $\mu = 0$. In figure 2 the expectation $E[T]$ is plotted versus the arrival and retrial rates. We have presented a surface which corresponds to $p = 6$ for the extreme case $\gamma = 1$.

In figures 3 and 4 we hold the same parameters p and γ. Then, we show the effect of the parameters α and λ on the partial coefficients of variation defined as $C^i = (M_2^i + M_1^i(1 - M_1^i))^{1/2}/M_1^i$, for $i = 0, 1$.

Table 1 illustrates the behaviour of the partial factorial moments M_k^0 as functions of p and the ratio α/λ. The system parameters are chosen to get $\gamma = 1$. The data correspond to the cases $p = 4, 6, 8, 10, 12$ and $\alpha/\lambda = 0.001, 0.01, 0.1, 1, 10$. The analysis of M_k^1 is similar due to the relationship (61).

A second set of numerical examples concerns with the linear retrial case $\alpha > 0$ and $\mu > 0$. We will continue assuming the extreme case, so $\rho = 1$.

The expected value $E[T]$ shown in figure 5 as a function of λ has two vertical asymptotes at the points $\lambda = 0$ and $\lambda = 4$. The second one is explained by the ergodicity condition $\lambda < \mu(p-1)$. We have presented the case $p = 5$, $\mu = 1$ and three curves which in decreasing order correspond to $\alpha = 0.5, 1, 2$. We have plotted the curves in the domain $\lambda \in (0.75, 4)$.

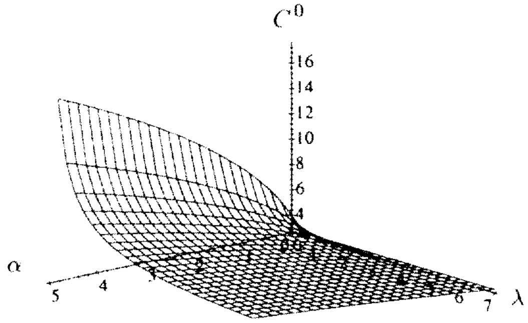

Figure 3: The effect of α and λ on C^0.

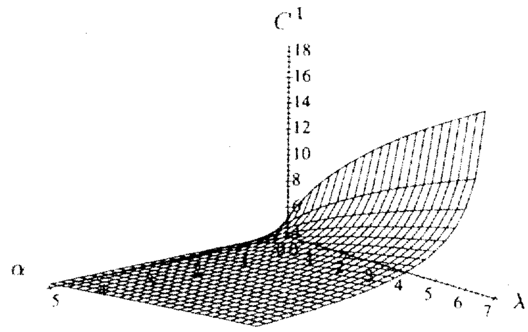

Figure 4: The effect of α and λ on C^1.

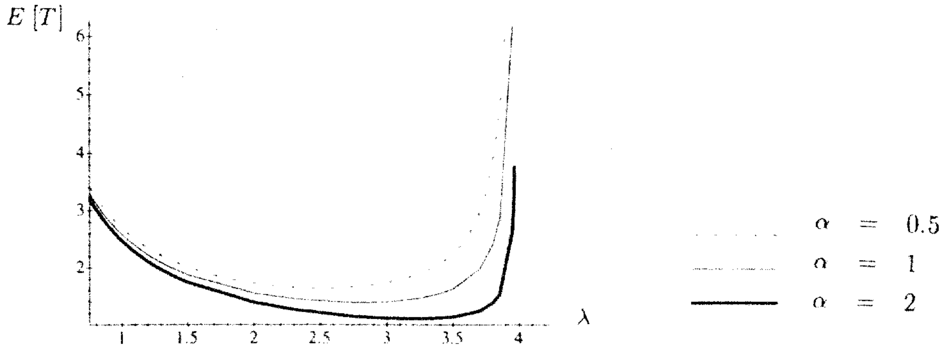

Figure 5: The effect of λ on $E[T]$.

p	$(\alpha/\lambda)\backslash k$	1	2	3	4	5	6	7	8	9	10
p=12	0.001	9.9809×10^{-2}	4.4359×10^{-2}	4.9904×10^{-2}	.11407	.47528	3.422	41.92	894.29	36219.	3.6219×10^{6}
	0.01	9.8119×10^{-2}	4.3609×10^{-2}	.04906	.11214	.46724	3.3641	41.21	879.15	35606.	3.5606×10^{6}
	0.1	8.3916×10^{-2}	3.7296×10^{-2}	4.1958×10^{-2}	9.5904×10^{-2}	.3996	2.8771	35.245	751.89	30451.	3.0451×10^{6}
	1	3.4286×10^{-2}	1.5238×10^{-2}	1.7143×10^{-2}	3.9184×10^{-2}	.16327	1.1755	14.4	307.2	12442.	1.2442×10^{6}
	10	4.9587×10^{-3}	2.2039×10^{-3}	2.4793×10^{-3}	5.6671×10^{-3}	2.3613×10^{-2}	.17001	2.0826	44.43	1799.4	1.7994×10^{5}

p	$(\alpha/\lambda)\backslash k$	1	2	3	4	5	6	7	8
p=10	0.001	.12476	7.1293×10^{-2}	.10694	.34221	2.1388	25.666	628.81	40244.
	0.01	.12267	7.0097×10^{-2}	.10515	.33646	2.1029	25.235	618.25	39568.
	0.1	.10504	6.0024×10^{-2}	9.0036×10^{-2}	.28812	1.807	21.609	529.41	33882.
	1	4.3103×10^{-2}	2.4631×10^{-2}	3.6946×10^{-2}	.11823	.73892	8.867	217.24	13903.
	10	.00625	3.5714×10^{-3}	5.3571×10^{-3}	1.7143×10^{-2}	.10714	1.2857	31.5	2016.0

p	$(\alpha/\lambda)\backslash k$	1	2	3	4	5	6
p=8	0.001	.16635	.13308	.29944	1.597	19.963	718.65
	0.01	.1636	.13088	.29448	1.5706	19.632	706.75
	0.1	.14035	.11228	.25263	1.3474	16.842	606.32
	1	5.7971×10^{-2}	4.6377×10^{-2}	.10435	.55652	6.9565	250.43
	10	8.4388×10^{-3}	6.7511×10^{-3}	.01519	.081013	1.0127	36.456

p	$(\alpha/\lambda)\backslash k$	1	2	3	4
p=6	0.001	.24954	.33272	1.4973	23.956
	0.01	.2455	.32733	1.473	23.568
	0.1	.21127	.28169	1.2676	20.282
	1	8.8235×10^{-2}	.11765	.52941	8.4706
	10	1.2931×10^{-2}	1.7241×10^{-2}	7.7586×10^{-2}	1.2414

p	$(\alpha/\lambda)\backslash k$	1	2
p=4	0.001	.49913	1.9965
	0.01	.4914	1.9656
	0.1	.42553	1.7021
	1	.18182	.72727
	10	2.7027×10^{-2}	.10811

Table 1. The effect of the system parameters on the factorial moments M_k^0 for a constant retrial queue with probabilities $q_k = k(k+p)^{-1}$.

M_k^0, $p=10$, $\alpha/\mu=1$									
$\lambda/\mu \diagdown k$	1	2	3	4	5	6	7	8	9
.5	1.4766×10^{-2}	6.5927×10^{-3}	8.0097×10^{-3}	2.1015×10^{-2}	.10714	1.0244	18.838	759.61	1.1697×10^{5}
1	3.1056×10^{-2}	1.7746×10^{-2}	2.6619×10^{-2}	8.5182×10^{-2}	.53239	6.3886	156.52	10017.	–
2.5	9.2171×10^{-2}	9.8994×10^{-2}	.26969	1.5927	19.82	575.66	66817.	–	–
6	.41424	2.0586	40.017	–	–	–	–	–	–
8	1.297	–	–	–	–	–	–	–	–

M_k^0, $p=8$, $\alpha/\mu=1$							
$\lambda/\mu \diagdown k$	1	2	3	4	5	6	7
.5	1.9296×10^{-2}	1.1764×10^{-2}	2.0661×10^{-2}	8.5211×10^{-2}	.78249	17.462	1606.0
1	4.1237×10^{-2}	3.2989×10^{-2}	7.4227×10^{-2}	.39588	4.9485	178.14	–
2.5	.13059	.21927	1.0718	14.739	915.65	–	–
6	1.0224	–	–	–	–	–	–

M_k^0, $p=6$, $\alpha/\mu=1$					
$\lambda/\mu \diagdown k$	1	2	3	4	5
.5	2.7786×10^{-2}	2.6679×10^{-2}	8.4466×10^{-2}	.81367	37.387
1	6.1224×10^{-2}	8.1633×10^{-2}	.36735	5.8776	–
2.5	.22399	.86837	21.089	–	–

M_k^0, $p=4$, $\alpha/\mu=1$			
$\lambda/\mu \diagdown k$	1	2	3
.5	4.9208×10^{-2}	.11081	1.7598
1	.11765	.47059	–
2.5	.80618	–	–

Table 2. The effect of the system parameters on the factorial moments M_k^0 for a linear retrial queue with probabilities $q_k = k(k+p)^{-1}$.

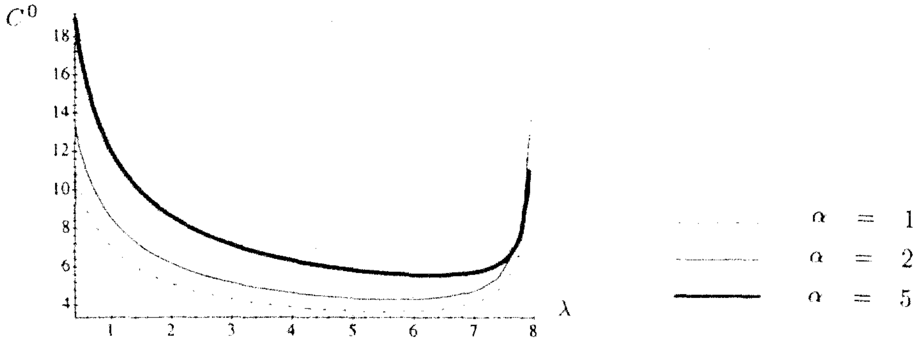

Figure 6: The effect of λ on C^0.

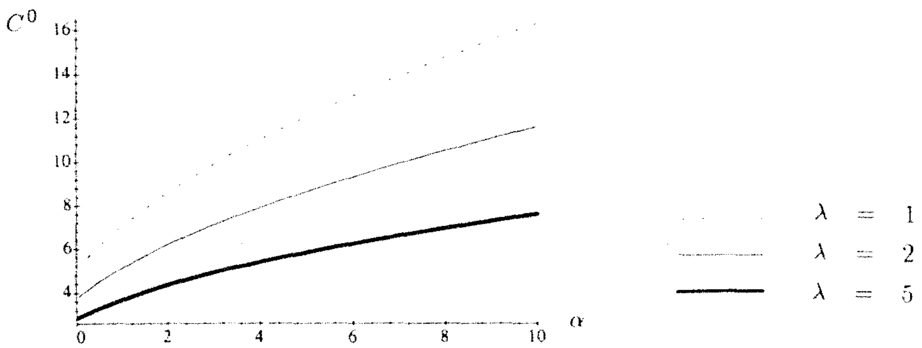

Figure 7: The effect of α on C^0.

The effect of the arrival rate and the retrial parameter α on the coefficient of variation C^0 is shown in figures 6 and 7, respectively. To this end, we have chosen $p = 10$ and $\mu = 1$. Then, in figure 6 we have presented three curves which correspond to $\alpha = 1, 2, 5$ and in figure 7 we consider the cases $\lambda = 1, 2, 5$.

Finally, table 2 illustrates the effect of the system parameters on the partial factorial moments M_k^0. Remember that M_k^0, $k \geq 1$, exists if and only if $\lambda < \mu(p - k)$. We have consider the case in which $\alpha/\mu = 1$ and $p = 4, 6, 8, 10$.

ACKNOWLEDGMENTS

The authors thank the referees for their suggestions that greatly improved the readability of this paper.

This research was supported by DGICYT through operating grant PB98-0837, the Complutense University through grant PR64/99-8501 and the European Commission through INTAS grant 96-0828.

REFERENCES

1. Abou-El-ata, M.O. and Hariri, A.M.A. (1995). Estimation and confidence intervals of the $M/M/2/N$ queue with balking and heterogeneity, *American Journal of Mathematics and Management Science*, 15, 35-55.

2. Artalejo, J.R. (1994). New results in retrial queueing systems with breakdown of the servers, *Statistica Neerlandica*, 48, 23-36.
3. Artalejo, J.R. (1995). A queueing system with returning customers and waiting line, *Operations Research Letters*, 17, 191-199.
4. Artalejo, J.R. and Gomez-Corral, A. (1997). Steady state solution of a single server queue with linear repeated requests, *Journal of Applied Probability*, 34, 223-233.
5. Choi, B.D., Shin, Y.W. and Ahn, W.C (1992). Retrial queues with collision arising from unslotted CSMA/CD protocol, *Queueing Systems*, 11, 335-356.
6. Çinlar, E. (1975). *Introduction to Stochastic Processes*, Prentice-Hall, Inc., Englewood Cliffs.
7. Cohen, J.W. (1957). Basic problems of telephone traffic theory and the influence of repeated calls, *Philips Telecommunication Review*, 18, 49-100.
8. Cox, D.R. and Isham, V. (1980). *Point Processes*, Chapman and Hall, London.
9. Falin, G.I. (1990). A survey of retrial queues, *Queueing Systems*, 7, 127-167.
10. Falin, G.I. and Artalejo, J.R. (1995). Approximations for multiserver queues with balking/ retrial discipline, *OR Spektrum*, 17, 239-244.
11. Falin G.I. and Templeton, J.G.C. (1997). *Retrial Queues*, Chapman and Hall, London.
12. Grassmann, W.K. (1974). The steady state behaviour of the $M/E_k/1$ queue, with state dependent arrival rates, *Infor*, 12, 163-173.
13. Haight, F.A. (1957). Queueing with balking, *Biometrika*, 44, 360-369.
14. Ikeda, Z. and Nishida, T. (1988). $M/G/1$ queue with balking, *Mathematica Japonica*, 33, 707-711.
15. Janssens, G.K. (1997). The quasi-random input queueing system with repeated attempts as a model for collision-avoidance star local area network, *IEEE Transactions on Communications*, 45, 360-364.
16. Khomichkov, I.I. (1995). Calculation of the characteristics of local area network with p-persistent protocol of multiple random access, *Automation and Remote Control*, 56, 208-218.
17. Kleinrock, L. (1975). *Queueing Systems, Vol. 1: Theory*, John Wiley and Sons, New York.
18. Kok, A.G. de (1984). Algorithmic methods for single server systems with repeated attempts, *Statistica Neerlandica*, 38, 23-32.
19. Krishna Kumar, B., Parthasarathy, P.R. and Sharafali, M. (1993). Transient solution of an M/M/1 queue with balking, *Queueing Systems*, 13, 441-448.
20. Kulkarni, V.G. (1995). *Modeling and Analysis of Stochastic Systems*, Chapman and Hall, London.
21. Martin, M. and Artalejo, J.R. (1995). Analysis of an $M/G/1$ queue with two types of impatient units, *Advances in Applied Probability*, 27, 840-861.
22. Schellhaas, H. (1983). Computation of the state probabilities in $M/G/1$ queues with state dependent input and state dependent service, *OR Spektrum*, 5, 223-228.
23. Schellhaas, H. (1986). Computation of the state probabilities in a class of semi-regenerative queueing models, in: *Semi-Markov Models: Theory and Applications* (Jansen, J. ed.), Plenum Press, 111-130.
24. Sennott, L.I., Humblet, P.A. and Tweedie, R.L. (1983). Mean drifts and the non-ergodicity of Markov chains, *Operations Research*, 31, 783-789.
25. Subba Rao, S. and Jaiswal, N.K. (1965). A queueing model with balking, reneging and limited servers' availability, *Opsearch*, 2, 31-43.
26. Tijms, H.C. (1994). *Stochastic Models: An Algorithmic Approach*, John Wiley and Sons, Chichester.
27. Yang, T. and Templeton, J.G.C. (1987). A survey on retrial queues, *Queueing Systems*, 2, 201-233.

J.R. Artalejo is Professor in the Faculty of Mathematics at the University Complutense of Madrid. He received the Ph.D. degree in Mathematics from University Complutense of Madrid in 1991. He has published his research papers in a variety of journals such as *Advances in Applied Probability, European Journal of Operational Research, Queueing Systems, OR Spektrum*, etc. He is Associate Editor of *Top* and Guest Editor of *Mathematical and Computer Modelling*. His research interests are in Queueing Theory and Stochastic Modelling of Communication Systems.

M.J. Lopez-Herrero received her Ph.D. degree from the Autonoma University of Madrid in 1994. Since 1987, she has been teaching in the School of Statistics at the University Complutense of Madrid. Her research interests include Harmonic Analysis, Queueing Theory and Telecommunication Systems.